much more difficult than is the computation of $\underline{v}$. The preceding example can illustrate this point if we note that the computation of g_1^c was trivial and resulted in the convex program P^c. The computation of $\bar{v}$, however, involved the solution of a nonconvex program (albeit a trivial one in this case) for certain values of u. The point is that $\underline{v}$ may still be desirable from a computational point of view, even though it does not bound v^* as accurately as does $\bar{v}$.

One case when the opposite is true occurs in the solution of separable, piecewise-linear, nonconvex problems. The Beale and Tomlin approach to such problems computes $\bar{v}$ implicitly by solving a large but structured linear program. The alternative would compute the convex envelope of each piecewise-linear function and then solve the resulting problem (itself a piecewise-linear problem). This approach actually involves more work and typically yields poorer estimates. A more detailed explanation of this point, together with a complete description of an efficient algorithm to solve such problems is available through the author.[1]

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Clustering a Data Array and the Traveling-Salesman Problem

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A clustering of a nonnegative $M \times N$ array is obtained by permuting its rows and columns. W. T. McCormick et al. [*Opns. Res.* 20, 993–1009 (1972)] measure the effectiveness of a clustering by the sum of all products of nearest-neighbor elements in the permuted array. This note points out that this clustering problem can be stated as two traveling-salesman problems.

A CLUSTERING of a nonnegative $M \times N$ array (a_{ij}) is obtained by permuting its rows and columns. In a recent article[3] the *measure of effectiveness* of a clustering was defined as the sum of all products of nearest-neighbor elements in the permuted array, and its maximization reduced to two separate and similar optimizations, one for the columns and the other for the rows. With the convention $\pi(0)=\pi(N+1)=a_{i0}=0$, the former problem is given by

$$\max_{\pi} \sum_{j=0}^{j=N} \sum_{i=1}^{i=M} a_{i\pi(j)} a_{i\pi(j+1)},$$

where $\pi=\{\pi(1), \pi(2), \cdots, \pi(N)\}$ runs over all $N!$ permutations of $\{1, 2, \cdots, N\}$.

It is stated in reference 3 that this subproblem may be rewritten as a *quadratic assignment problem* (QAP). More precisely, it is a *traveling-salesman problem* (TSP) with cities $0, 1, \cdots, N$ and distances

$$c_{jk} = -\sum_{i=1}^{i=M} a_{ij} a_{ik}. \qquad (j, k=0, 1, \cdots, N)$$

In general, the clustering problem for a p-dimensional array can be stated as p TSPs, and it may be attacked by any optimal or suboptimal algorithm for the TSP. In fact, the proposed *bond energy algorithm*[3] is a simple suboptimal TSP-method (cf. reference 4, p. 76). It performs well for this type of problem: in reference 3, the solution to the airport example is 0.5 percent below the optimum, and for the marketing example the optimum is found. One might supplement this method with one of the fast clean-up procedures.[1,2,5] Optimal algorithms are out of the question if the array is large, but, fortunately, informative patterns seem to emerge without rigorous optimization.

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